SISTEMAS

$$\Sigma: \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases} \quad a_{ij}, b_j \in K \quad \Bigg| \quad \Sigma: A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Conjunto de soluciones

$$\underbrace{\Sigma(K)}_{} = \{(a_1, \dots, a_n) \in K^n : a_{11}a_1 + \dots + a_{n}a_{1n} = b_1, \dots, a_{m1}a_1 + \dots + a_{mn}a_n = b_m\}$$

i) $¿\Sigma(K) \neq \emptyset$?

ii) $\Sigma(K) \neq \emptyset \Longrightarrow ¿\Sigma(K)$?

} Lo que hay siempre que comprobar

$b_1 = \dots = b_m = 0 \Longrightarrow$ Sistema homogéneo

$$\Downarrow$$

$$A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

$(A|b) \in Mat_{m \times (n+1)}$

→ Matriz ampliada

Teorema

Si Σ es homogéneo, entonces $\Sigma(K)$ es un subespacio vectorial de K^n de dimensión $n - rg(A)$

!!

Dem.

$$\Sigma: A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \qquad v = (x_1, \dots, x_n)_{B_c} \Longrightarrow \emptyset(v) = (x_1', \dots, x_m')$$

$$\emptyset: K^n \longrightarrow K^m \qquad \begin{pmatrix} x_1' \\ \vdots \\ x_m' \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Longrightarrow \Sigma(K) = Ker(\emptyset)$$

$$M_{B_c, B_c}(\emptyset) = A \qquad \Longrightarrow \Sigma(K) < K^n,$$

$$\dim \Sigma(k) = n - rg(\emptyset) = n - rg(A)$$

$$\Sigma: \begin{cases} a_{11}x_1 + \dots + a_{1r}x_r + \dots + a_{1n}x_n = 0 \\ \vdots \\ a_{r1}x_1 + \dots + a_{rr}x_r + \dots + a_{rn}x_n = 0 \\ \vdots \\ \cancel{a_{m1}x_1} + \dots + \cancel{a_{mr}x_r} + \dots + \cancel{a_{mn}x_n} = 0 \end{cases}$$

} Las l.i.

} Combinación lineal

Variables independientes

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} -a_{1\,r+1}x_{r+1} - \dots - a_{1n}x_n \\ \vdots \\ -a_{r\,r+1}x_{r+1} - \dots - a_{rn}x_n \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}^{-1} \cdot \begin{pmatrix} -a_{1\,r+1}x_{r+1} - \dots - a_{1n}x_n \\ \vdots \\ -a_{r\,r+1}x_{r+1} - \dots - a_{rn}x_n \end{pmatrix}$$

Ej.:

$$\Sigma:\begin{cases} x+y-2z+t=0 \\ x-2y+z-2t=0 \\ -2x+y-z+t=0 \end{cases} \sim \Sigma'\begin{cases} x+y-2z+t=0 \\ -3y+3z-3t=0 \\ 3y-3z+3t=0 \end{cases} \sim \Sigma''\begin{cases} x+y-2z+t=0 \\ 3y+3z-3t=0 \\ 0=0 \end{cases}$$

$$A''=\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & -3 \end{matrix}} & \begin{matrix} -2 & 1 \\ 3 & -3 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{pmatrix} \qquad \Sigma(K)=\Sigma''(K)<K^4,\ \dim\Sigma(K)=4-rg(A'')=4-2=2$$

$$\sim \Sigma''':\begin{cases} x \quad -z=0 \\ -3y+3z-3t=0 \end{cases} \quad A'''=\begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & -3 \end{matrix}} & \begin{matrix} -1 & 0 \\ 3 & -3 \end{matrix} \end{pmatrix} \Longrightarrow \Sigma(K)\begin{cases} x=z \\ y=z-t \end{cases} \Rightarrow \underline{\Sigma(K)=\{(x,y,z,t)\in K^4 \,/\, x=z,\ y=z-t\}}$$

$$\{\underbrace{(0,1,1,0),\ (0,-1,0,1)}_{\text{Elegidas}}\}\ \text{base de } \Sigma(K)$$

$$rg(A)=rg\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 1 & -2 \end{matrix}} & \begin{matrix} -2 & 1 \\ 1 & -2 \end{matrix} \\ \begin{matrix} -2 & 1 \end{matrix} & \begin{matrix} 1 & 1 \end{matrix} \end{pmatrix}=2 \qquad P=\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$F_3=-(F_1+F_2)$ $\quad C_3=-(C_1+C_2)$ $\quad C_4=C_2$ $\quad$ { Vemos que son l.d.

$$\Downarrow$$

$$\Sigma:\begin{cases} x+y-2z+t=0 \\ x-2y+z-2t=0 \end{cases} \Longleftrightarrow P\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2z-t \\ -z+2t \end{pmatrix}$$

la tercera sobra (no aporta nada)

$$\Downarrow$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=P^{-1}\begin{pmatrix} 2z-t \\ -z+2t \end{pmatrix}$$

$$P^{-1}=\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2z-t \\ -z+2t \end{pmatrix}=$$

$$=\frac{1}{3}\begin{pmatrix} 2(2z-t)+(-z+2t) \\ (2z-t)-1(-z+2t) \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} 3z \\ 3z-3t \end{pmatrix}=\underline{\begin{pmatrix} z \\ z-t \end{pmatrix}}$$

Nunca es un espacio vectorial

$$\Sigma_b : A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\Sigma_0 : A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Sistema homogéneo asociado

Teorema de Rouche - Frobenius

i) $\Sigma_b(K) \neq \emptyset \iff rg(A) = rg(A|b)$

ii) Si $\Sigma_b(K) \neq 0$, entonces $\Sigma_b(K) = \{(a_1, \dots, a_n) + (c_1, \dots, c_n) : (c_1, \dots, c_n) \in \Sigma_0(K)\}$, donde $(a_1, \dots, a_n) \in \Sigma_b(K)$

Dem.

$\emptyset : \underset{\substack{B_c \\ \parallel \\ \{e_1, \dots, e_n\}}}{K^n} \longrightarrow \underset{\substack{B'_c \\ \parallel \\ \{e'_1, \dots, e'_n\}}}{K^m}$, $M_{B_c, B'_c}(\emptyset) = A$, $v = (x_1, \dots, x_n)_{B_c}$, $\emptyset(v) = (x'_1, \dots, x'_m)_{B'_c}$

$$\implies \begin{pmatrix} x'_1 \\ \vdots \\ x'_m \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Sigma_b(K) \neq \emptyset \iff (b_1, \dots, b_m)_{B'_c} \in Im(\emptyset) \iff b_1e'_1 + \dots + b_me'_m \in L(\emptyset(e_1), \dots, \emptyset(e_n))$$

$$\iff L(b_1e'_1 + \dots + b_me'_m, \emptyset(v_1), \dots, \emptyset(v_n)) = Im(\emptyset)$$

$$\iff \dim L(\underbrace{b_1e'_1 + \dots + b_me'_m, \emptyset(v_1), \dots, \emptyset(v_n)}_{n+1 \text{ vectores columna}}) = \dim Im(\emptyset) = rg\,\emptyset = rg\,A$$

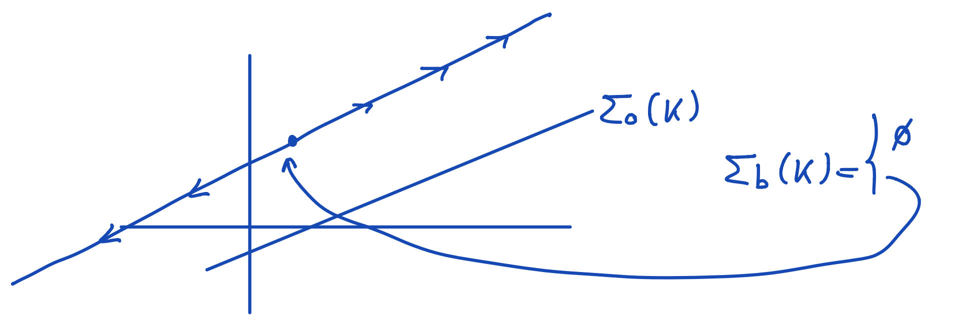
$$\parallel$$

$$\iff rg(A|b) = rg(A)$$

ii) Sea $(a_1, \dots, a_n) \in \Sigma_b(K)$

Sea $(a'_1, \dots, a'_n) \in \Sigma_b(K)$ Otra cualquiera

$$(\underset{\substack{\parallel \\ c_1}}{a'_1 - a_1}, \dots, \underset{\substack{\parallel \\ c_n}}{a'_n - a_n}) \in \Sigma_0(K) \implies (a'_1, \dots, a'_n) = \underbrace{(a_1, \dots, a_n)}^{\in \Sigma_b(K)} + \overbrace{(c_1, \dots, c_n)}^{\in \Sigma_0(K)}$$



Método de Cramer

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases} \qquad rg(A)=3 \Longrightarrow |A| \neq 0$$

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{|A|}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{|A|}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{|A|}$$

Dem.

Sean x, y, z las soluciones únicas. Los interpretamos como elementos de K

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1y & b_1 & c_1 \\ b_2y & b_2 & c_2 \\ b_3y & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} c_1z & b_1 & c_1 \\ c_2z & b_2 & c_2 \\ c_3z & b_3 & c_3 \end{vmatrix} =$$

$$= x \cdot |A| + y \cdot 0 + z \cdot 0 = x \cdot |A|$$

Análogo para y, z. (Demostración válida para cualquier $n \times n$)

rango = tamaño máximo de un menor

$rg(A) = r$, $A \in Mat_{m \times n}(K)$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1r} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} & \cdots & a_{rn} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mr} & \cdots & a_{mn} \end{pmatrix} \Big\} \text{Vectores l.i.} \Longrightarrow rg\begin{pmatrix} a_{11} & \cdots & a_{1r} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} & \cdots & a_{rn} \end{pmatrix} = r$$

l.i. (ya que hay r columnas indep)

$$\Longrightarrow rg\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} = r$$

Supongamos $t > r \Longrightarrow$ No habrá submatrices que den mayor rango